Young Women in Algebraic Geometry

Local and adelic Hecke algebra isomorphisms

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Adelic and local L^1-isomorphisms

Let K be a number field or a non-archimedean local field of characteristic zero. Let G be a linear algebraic group over $\mathbb{Q}$.

Definition. The **finite-adelic resp. local Hecke algebra** is

$$\mathcal{H}_G(K) = \begin{cases} C_c^\infty(G(\mathbb{A}_{K,f}), \mathbb{R}) & \text{if } K \text{ is global} \\ C_c^\infty(G(K), \mathbb{C}) & \text{if } K \text{ is local.} \end{cases}$$

The (left) invariant Haar measure induces an L^1-norm on the Hecke algebras. An L^1**-isomorphism** between Hecke algebras is an algebra isomorphism which is an isometry for the L^1-norm.

Theorem A. Let K and L be two number fields, respectively two non-archimedean local fields of characteristic zero. There is an L^1-isomorphism of Hecke algebras $\mathcal{H}_G(K) \cong_{L^1} \mathcal{H}_G(L)$ if and only if there is

$$\begin{cases} \text{a ring isomorphism } \mathbb{A}_K \cong \mathbb{A}_L & \text{if } K \text{ is global} \\ \text{a field isomorphism } K \cong L & \text{if } K \text{ is local.} \end{cases}$$

N.B. These isomorphisms are topological.

Proof. Denote

$$G(K) = \begin{cases} G(\mathbb{A}_{K,f}) & \text{if } K \text{ is global} \\ G(K) & \text{if } K \text{ is local.} \end{cases}$$

Stone-Weierstrass yields that $\mathcal{H}_G(K)$ is dense in $C_0(G(K))$, hence in $C_c(G(K))$, hence in $L^1(G(K))$. So $\mathcal{H}_G(K) \cong_{L^1} \mathcal{H}_G(L)$ induces an isometry $L^1(G(K)) \cong_{L^1} L^1(G(L))$ of group algebras. Now $L^1(G(K)) \cong_{L^1} L^1(G(L))$ implies that $G(K) \cong G(L)$ by results of Wendel [4].
For K and L local, it is a classical result that $G(K) \cong G(L)$ implies that $K \cong L$.
For K and L global, this follows from Theorem B. $\square$

Adelic point group isomorphisms.

Let K be a number field and G a linear algebraic group over $\mathbb{Q}$.

Definition. G is **fertile** for K if it is K-split and its connected component of the identity is *not* a direct product $T \times U$ of a torus and a unipotent group.

Theorem B. Let K and L be number fields and let $G/\mathbb{Q}$ be fertile for K and L. There is a topological group isomorphism $G(\mathbb{A}_{K,f}) \cong G(\mathbb{A}_{L,f})$ if and only if there is a topological ring isomorphism $\mathbb{A}_K \cong \mathbb{A}_L$.

Proof.
Step 1: Let U be a fixed maximal unipotent subgroup of G. Then any maximal divisible subgroup $\mathbb{D}$ of $G(\mathbb{A}_{K,f})$ is conjugate to $U(\mathbb{A}_{K,f})$, characterising $U(\mathbb{A}_{K,f})$ group theoretically inside $G(\mathbb{A}_{K,f})$.
Step 2:

- Let $\mathbb{N} = N_{G(\mathbb{A}_{K,f})}(\mathbb{D})$ be the normaliser of $\mathbb{D}$.
- Let $\mathbb{V} = [\mathbb{N}, \mathbb{D}]/[\mathbb{D}, \mathbb{D}] \leq \mathbb{D}^{\mathrm{ab}}$.
- Let $\mathbb{T} = \mathbb{N}/\mathbb{D}$. It acts on $\mathbb{V}$ through say ℓ distinct non-trivial characters.

Then $Z(\mathrm{End}_{\mathbb{T}}(\mathbb{V})) \cong \mathbb{A}_{K,f}^\ell$.
Step 3: Forming quotients $\mathbb{A}_{K,f}^\ell/\mathfrak{m}$ by maximal ideals yields the multiset $\{K_\mathfrak{p} : \mathfrak{p} \in \mathcal{M}_{K,f}\}$, where $\mathcal{M}_{K,f}$ is the set of finite places of K.
Step 4: Doing this construction for both K and L proves that K and L are locally isomorphic, i.e. $\mathbb{A}_{K,f} \cong \mathbb{A}_{L,f}$, i.e. $\mathbb{A}_K \cong \mathbb{A}_L$, by results of Klingen [3]. $\square$

Remark. The proof goes through for abstract isomorphisms instead of topological isomorphisms.

Local Morita equivalences

Let K be a non-archimedean local field of characteristic zero and let $G = \mathrm{GL}_2$.

Theorem C. There is a Morita equivalence

$$\mathcal{H}_G(K) \sim_M \left(\bigoplus_{\mathbb{N}} \mathbb{C}[T, T^{-1}] \right) \oplus \left(\bigoplus_{\mathbb{N}} \mathbb{C}[X, X^{-1}, Y, Y^{-1}] \right) \oplus \left(\bigoplus_{\mathbb{N}} \mathbb{C}[\mathbb{Z}^2 \rtimes S_2] \right).$$

Proof. We use the Bernstein decomposition

$$\mathcal{H}_G(K) = \bigoplus_{s \in \mathcal{B}(G(K))} \mathcal{H}_G^s(K)$$

where

$$\mathcal{H}_G^s(K) = \mathcal{H}_G(K) * e_\rho * \mathcal{H}_G(K) \sim_M e_\rho * \mathcal{H}_G(K) * e_\rho$$

for some idempotent e_ρ depending on a representation $\rho \colon H \to \mathrm{End}_{\mathbb{C}}(W)$ of a compact open subgroup H of $G(K)$, and

$$e_\rho * \mathcal{H}_G(K) * e_\rho \cong \mathcal{H}(G(K), \rho) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(W),$$

where $\mathcal{H}(G(K), \rho)$ is the ρ-spherical or intertwining algebra for ρ. Now,

$$\mathcal{H}(G(K), \rho) \cong \begin{cases} \mathbb{C}[T, T^{-1}] & \text{if } \rho \text{ is supercuspidal} \\ \mathbb{C}[X, X^{-1}, Y, Y^{-1}] & \text{if } K \text{ is a non-special principal series} \\ \mathbb{C}[\mathbb{Z}^2 \rtimes S_2] & \text{if } \rho \text{ is a special rep. or finite-dim.,} \end{cases}$$

each of which occurs countably infinitely many times. $\square$

Corollary D. Let K and L be any non-arch. local fields of char. zero. Then

$$\mathcal{H}_G(K) \sim_M \mathcal{H}_G(L).$$

Remarks and references

Remark 1. Let K and L be number fields. The additive groups $(\mathbb{A}_K, +)$ and $(\mathbb{A}_L, +)$ are isomorphic if and only if $[K : \mathbb{Q}] = [L : \mathbb{Q}]$. However, if this isomorphism is *local*, then K and L are arithmetically equivalent.

Remark 2. Let K and L be imaginary quadratic number fields, different from $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-2})$. Then the multiplicative groups $\mathbb{A}_K^*$ and $\mathbb{A}_L^*$ are isomorphic. Again, if this isomorphism is *local*, then K and L are arithmetically equivalent.

Remark 3. Remarks 1 and 2 show why we need fertility in Theorem B.

References

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